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Thermal impedance and transient temperature due to a spot of heat on a half-space

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ABSTRACT

Analytical solutions are proposed in this paper to calculate the thermal impedance and transient temperature in a semi-infinite body subjected to a heat spot. Two cases of uniform and non-uniform heat flux are considered. These solutions are developed using integral transforms and are given in exact expression forms, without any restrictive hypothesis. They include special functions such as Bessel, Struve and hypergeometric functions. Many softwares to treat these functions are available (eg. Maple, Mathematica and others). The solutions are validated through comparisons with available models treating particular cases. The transient temperatures and impedances are presented for different spatial distributions of the heat flux dissipated by the spot.

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1. Introduction

Spot heating occurs in many technical domains such as electronics, welding, laser beam, etc. In all these applications, the temperature field should be known precisely. In the case of electronic components, the temperature should be not attain a critical limit which may damage it. For welding, or laser treatment, the temperature level should be controlled to obtain successful results. Many studies are developed in the literature to treat spot heating. Most of these studies are limited to steady state cases [1–7]. Such cases do not take into consideration the spontaneous variation of the heating power. A limited number of studies are concerned with transient cases. Some authors have given approximations for the thermal impedance [8–10] and the transient temperature [11]. These studies concern only the case of uniform heat source. Some works concerning the study of thermal resistance and temperatures for moving contacts have been developed [12–15].

Exact analytical solutions – based on the use of integral transforms – are presented in this paper. These solutions express the thermal impedance and the transient temperature evolution in a body subjected to spot heating. The results are compared to particular cases available in the literature. The impedance and transient temperature are analysed and presented for different physical and geometric parameters.

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2. General solution

The problem is represented on Fig. 1. A semi-infinite medium, initially at zero reference temperature, is suddenly subjected to a heat spot of radius a and density φ.

The governing equations of heat diffusion in the solid are:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

$$T(r,z,0)=0 \tag{2}$$

$$-\lambda\left(\frac{\partial T}{\partial z}\right)_{z=0}=\begin{cases}\varphi & (r\le a)\\ 0 & (r>a)\end{cases} \text{ and } (T)_{z\to\infty}=0 \tag{3}$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=0}=0 \text{ and } (T)_{r\to\infty}=0 \tag{4}$$

The following Laplace and infinite Hankel transforms are applied to equations (1)–(4):

$$\overline{T}=L\{T\}=\int_0^\infty Te^{-pt}dt$$

$$\tilde{\overline{T}}=\int_0^\infty rJ_0(\beta r)\overline{T}dr \tag{5}$$

Nomenclature

a	Radius of the spot, m
F	Hypergeometric function
I_n	Modified Bessel function of the first kind of order n
J_n	Bessel function of the first kind of order n
L	Laplace operator
L_n	Modified Struve function of order n
p	Laplace variable, s^{-1}
R	Thermal resistance, m^2 K W^{-1}
r, z	Cylindrical coordinate, m
T	Temperature rise, °C
t	Time, s
Z	Thermal impedance, m^2 K W^{-1}

Greek Symbols

α	Thermal diffusivity, m^2 s^{-1}
β	Hankel variable, m^{-1}
φ	Heat flux density, W m^{-2}
λ	Thermal conductivity, W m^{-1} K^{-1}

Subscript

1, 2	Case 1 or 2
a	Apparent area
av	Average
c	Contact (or Constriction)

Superscript

ap	Approximated
*	Dimensionless quantity
$\sim$	Hankel transform
$-$	Laplace transform

Where: L is the Laplace operator, p the Laplace variable, J_0 the Bessel function of the first kind of order zero and β the Hankel variable.

The transformed equations become:

$$\frac{d^2\tilde{\overline{T}}}{dz^2} - \left[\beta^2 + p/\alpha\right]\tilde{\overline{T}} = 0 \tag{6}$$

$$-\lambda\left(\frac{d\tilde{\overline{T}}}{dz}\right)_{z=0} = \tilde{\overline{\varphi}} \text{ and } (\tilde{\overline{T}})_{z\to\infty} = 0 \tag{7}$$

The solution of equation (6) with the boundary conditions (7) can be written in the following form:

$$\tilde{\overline{T}} = \frac{\tilde{\overline{\varphi}}}{\lambda}\frac{e^{-\sqrt{\beta^2+p/\alpha}z}}{\sqrt{\beta^2+p/\alpha}} \tag{8}$$

The following Hankel and Laplace inverse transforms can be used to obtain T:

$$\overline{T} = \int_0^\infty \beta J_0(\beta r)\tilde{\overline{T}}\, d\beta$$

$$T = L^{-1}\{\overline{T}\} \tag{9}$$

Where L^{-1} is the Laplace inverse operator.

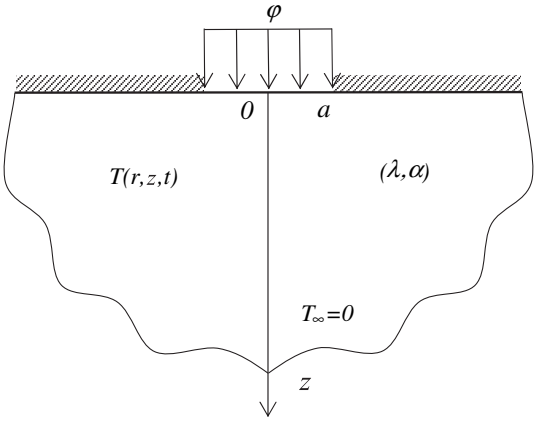

Fig. 1. The physical configuration.

The thermal impedance caused by the spot of heat can be calculated using the following relation:

$$\overline{Z} = \frac{\overline{T}_{av} - \overline{T}_a}{\overline{\varphi}_{av}\pi a^2} \tag{10}$$

Where $\overline{T}_{av}$ is the average temperature of the heat spot area, $\overline{T}_a$ the average temperature of the body apparent area ($\overline{T}_a = 0$, since the reference temperature is zero) and $\overline{\varphi}_{av}$ the average heat flux on the spot area with:

$$\begin{aligned} \overline{T}_{av} &= \frac{1}{\pi a^2}\int_0^a \overline{T}(r, z=0)2\pi r\, dr \\ \overline{T}_a &= 0 \\ \overline{\varphi}_{av} &= \frac{1}{\pi a^2}\int_0^a \overline{\varphi}\, 2\pi r\, dr \end{aligned} \tag{11}$$

So:

$$\overline{Z} = \frac{2}{\pi\lambda a^3\overline{\varphi}_{av}}\int_0^\infty \frac{\tilde{\overline{\varphi}}J_1(\beta a)}{\sqrt{\beta^2+p/\alpha}}d\beta \tag{12}$$

In what follows, the explicit expression of the integral (12) is given according to the flux φ imposed by the spot.

3. Exact solution for frequent cases

Many cases can be considered, relative to the flux spatial and temporal distribution. Two cases, usually applied in practice, will be considered in this study:

- Case 1 – uniform flux: $\varphi = \varphi_1$ (φ_1 is a constant).
- Case 2 – non-uniform flux: $\varphi = \varphi_2/\sqrt{1-(r/a)^2}$ (φ_2 is a constant).

In the literature, the case 2 is commonly called 'equivalent isothermal spot'. It corresponds to a uniform spot temperature in the steady state regime.

In what follows, solution details for the temperature and the thermal impedance are presented using the following dimensionless parameters:

$$p^* = pa^2/\alpha,\ \overline{Z}^* = \overline{Z}\lambda a,\ t^* = \alpha t/a^2.$$

3.1. Case 1 – uniform heat flux $\varphi = \varphi_1$ (φ_1 is a constant)

Applying Laplace and Hankel integral transforms to the flux yields the following expression for $\bar{\tilde{\varphi}}$:

$$\bar{\tilde{\varphi}}_1 = \varphi_1 \frac{aJ_1(\beta a)}{p\beta} \tag{13}$$

where J_1 is the Bessel function of first kind of order one.

Combining equations (13) and (12) leads to the expression of thermal impedance, such that:

$$\bar{Z}_1^* = \frac{2}{\pi}\int_0^\infty \frac{J_1^2(\beta a)}{\beta\sqrt{(\beta a)^2+p^*}}\,d\beta \tag{14}$$

The integration in the above equation can be written under the following form [16]:

$$\bar{Z}_1^* = \frac{1}{\pi p^*}\left[\sqrt{p^*} + L_1\left(2\sqrt{p^*}\right) - I_1\left(2\sqrt{p^*}\right)\right] \tag{15a}$$

or:

$$\bar{Z}_1^* = \frac{1}{\pi}\left[\frac{8}{3\pi}F\left(1;\frac{5}{2};\frac{3}{2};p^*\right) - \frac{1}{2}\sqrt{p^*}F(1;2,3;p^*)\right] \tag{15b}$$

Where L_1 is the modified Struve function of order one, I_1 the modified Bessel function of first kind of order one and F the hypergeometric function. All these special functions are available in softwares such as Maple, Mathematica and others.

When $p = 0$ (ie. when $t \to \infty$) expression (15b) yields the steady state thermal constriction resistance: $R_{c,1} = \bar{Z}_1(p=0) = 8/(3\pi^2\lambda a) \cong 0.2702/(\lambda a)$.

An approximation for equation (14) has been proposed in the literature by considering the asymptotic cases: $p = 0$ and $p \to \infty$ [8,9]. The approximated solution $\bar{Z}_1^{ap}$ is given by:

$$\bar{Z}_1^{ap} = \frac{8/(3\pi^2\lambda a)}{1+8\sqrt{pa^2/\alpha}/(3\pi)}$$

$$\bar{Z}_1^{ap^*} = \frac{8/(3\pi^2)}{1+8\sqrt{p^*}/(3\pi)} \tag{16}$$

The evolution of $\bar{Z}_1^*$ – equations (15a) or (15b) – and $\bar{Z}_1^{ap^*}$ – equation (16) – is given on Fig. 2. The maximum difference between the two solutions is about 11%.

The transient surface temperature can be calculated from equation (8) after replacing $\bar{\tilde{\varphi}}$ by its expression given by (13) as

$$T_1(r,0,t) = \frac{\varphi_1 a}{\lambda}\int_0^\infty J_1(\beta a)J_0(\beta r)L^{-1}\left[\frac{1}{p\sqrt{\beta^2+p/\alpha}}\right]d\beta \tag{17}$$

Putting $t^* = t\alpha/a^2$ and $T^* = T/(\varphi_1 a/\lambda)$, we obtain:

(i) The temperature at the spot centre ($r = 0$):

$$T_1^*(0,0,t^*) = erfc\left(\frac{1}{2\sqrt{t^*}}\right) + 2\sqrt{\frac{t^*}{\pi}}\left[1-e^{-1/(4t^{*2})}\right] \tag{18}$$

(ii) The temperature at the external contour of the spot ($r = a$)

$$T_1^*(r=a,0,t^*) = \frac{2}{\pi} + \sqrt{\frac{t^*}{\pi}} - \sqrt{\frac{1}{\pi t^*}}\,e^{-\frac{1}{2t^*}} \times\left[(1+t^*)I_0\left(\frac{1}{2t^*}\right) + I_1\left(\frac{1}{2t^*}\right)\right] \tag{19}$$

(iii) The average value:

$$T_{1,av}^*(t^*) = \frac{8}{3\pi} + 2\sqrt{\frac{t^*}{\pi}} - 2\sqrt{\frac{t^*}{\pi}}\,e^{-\frac{1}{2t^*}}\left[\left(1+\frac{2}{3t^*}\right)I_0\left(\frac{1}{2t^*}\right) + \left(\frac{1}{3}+\frac{2}{3t^*}\right)I_1\left(\frac{1}{2t^*}\right)\right] \tag{20}$$

All the temperatures increase rapidly and attain their steady values in about $t^* \cong 10$ (Fig. 3). The asymptotic values of these temperatures can be calculated from equations (18)–(20) with: $t^* \to \infty$. We get:

(i) $T_1^*(0,0,\infty) = 1$

(ii) $T_1^*(r=a,0,\infty) = 2/\pi \cong 0.6366$

(iii) $T_{1,av}^*(\infty) = 8/(3\pi) \cong 0.8488$.

Beck et al. [11] have proposed some approximations to calculate the transient temperature in the case of a uniform heat source. Table 1 compares the average temperature of the spot between the approximation given by these authors for short time and the exact solution given by equation (20). The results are in excellent agreement.

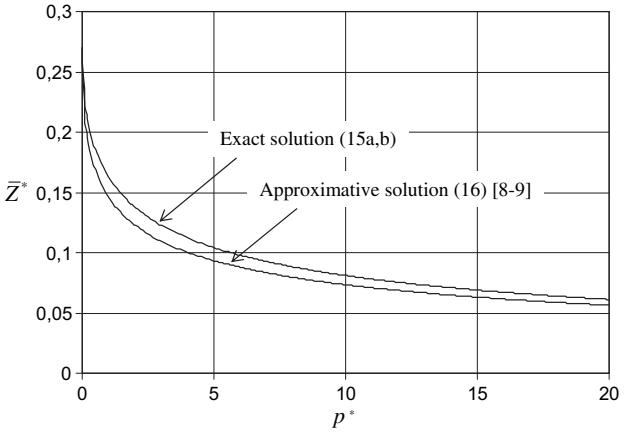

Fig. 2. Evolution of $\bar{Z}_1$ as a function of the Laplace variable p.

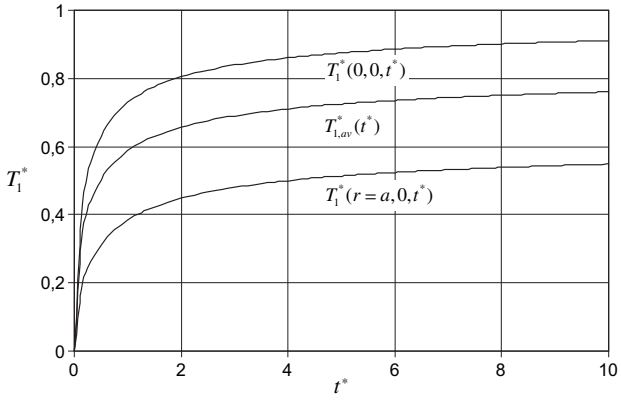

Fig. 3. Evolution of dimensionless surface temperature as a function of time (Case 1).

Table 1
Comparison of average temperatures between the exact solution (20) and the approximation given in the Ref. [11] for short time.

t^*	$T^*_{1,av}$		t^*	$T^*_{1,av}$	
	[11]	Equation (20)		[11]	Equation (20)
0.01	0.10647976	0.10647969	0.06	0.23850180	0.23848918
0.02	0.14687698	0.14687651	0.07	0.25439491	0.25437495
0.03	0.17641618	0.17641458	0.08	0.26877428	0.26874463
0.04	0.20034346	0.20033969	0.09	0.28192056	0.28187854
0.05	0.22069115	0.22068382	0.1	0.29403820	0.29398086

3.2. Case 2 – non-uniform heat flux $\varphi = \varphi_2/\sqrt{1-(r/a)^2}$ (φ_2 is a constant)

This distribution is called 'equivalent isothermal spot' since it corresponds to the steady state uniform temperature.

The application of Laplace and Hankel transforms to the flux distribution over the spot area yields $\tilde{\bar{\varphi}}_2$ under the following form:

$$\tilde{\bar{\varphi}}_2 = \varphi_2 \frac{a\sin(\beta a)}{p\beta} \tag{21}$$

Combining equations (21) and (12) leads to the dimensionless expression of the thermal impedance $\bar{Z}^*_2 = \bar{Z}_2\lambda a$ as a function of $p^* = pa^2/\alpha$:

$$\bar{Z}^*_2 = \frac{1}{\pi}\int_0^\infty \frac{J_1(\beta a)\sin(\beta a)}{\beta\sqrt{(\beta a)^2+p^*}}\,d\beta \tag{22}$$

$$\bar{Z}^*_2 = \frac{1}{4}F\left(\frac{5}{4},\frac{3}{4};1,2,\frac{3}{2};p^*\right) - \frac{\sqrt{p^*}}{2\pi}F\left(1,\frac{5}{4},\frac{7}{4};2,\frac{3}{2},\frac{3}{2},\frac{5}{2};p^*\right) \tag{23}$$

Fig. 4 shows the evolution of $\bar{Z}^*_2$ and $\bar{Z}^*_1$ with the dimensionless Laplace variable p^*. It can be seen that $\bar{Z}^*_2$ is systematically less than $\bar{Z}^*_1$. When $p\to 0$ (ie. When $t\to\infty$: which corresponds to the steady state situation) equation (23) yields the following known value: $R^*_{c,2} = R_{c,2}\lambda a = \bar{Z}^*_2(p^*=0) = 1/4 = 0.25$. This value is slightly less than that of uniform flux case, previous given as: $R^*_{c,1} = R_{c,1}\lambda a = \bar{Z}^*_1(p^*=0) = 8/(3\pi^2) = 0.2702$. The difference between the two cases is about 8%.

The dimensionless transient temperatures, $T^*_2 = T_2/(\varphi_2 a/\lambda)$, at the spot surface can be written as follows:

(i) for $r = 0$

$$T^*_2(0,0,t^*) = \frac{\pi}{2} - \frac{1}{\sqrt{\pi t^*}}F\left(1,\frac{1}{2},\frac{3}{2},\frac{3}{2};-\frac{1}{4t^*}\right) \tag{24}$$

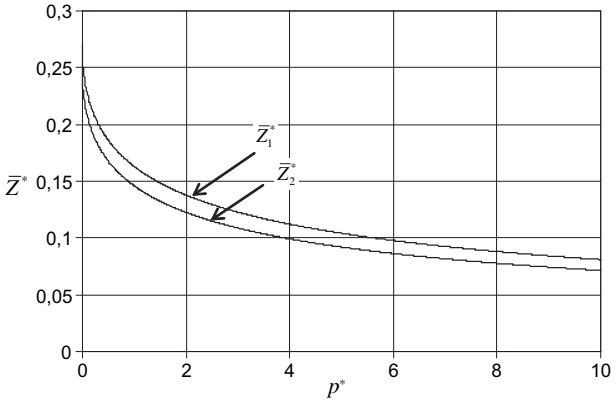

Fig. 4. Evolution of $\bar{Z}_1$ and $\bar{Z}_2$ as a function of the Laplace variable p.

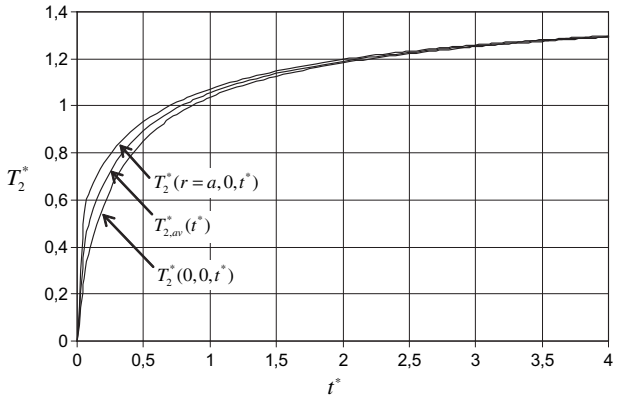

Fig. 5. Evolution of dimensionless surface temperature as a function of time (Case 2).

(ii) for $r = a$

$$T^*_2(a,0,t^*) = \frac{\pi}{2} - \frac{1}{\sqrt{\pi t^*}}F\left(\frac{1}{2},\frac{3}{4},\frac{5}{4};\frac{3}{2},\frac{3}{2},\frac{3}{2};-\frac{1}{t^*}\right) \tag{25}$$

(iii) the average value

$$T^*_{2,\mathrm{av}}(t^*) = \frac{\pi}{2} - \frac{1}{\sqrt{\pi t^*}}F\left(1,\frac{1}{2},\frac{7}{4},\frac{5}{4};2,\frac{3}{2},\frac{3}{2},\frac{5}{2};-\frac{1}{t^*}\right) \tag{26}$$

As shown on Fig. 5, all temperatures increase rapidly to attain a unique value: $T^*_2(0,0,\infty) = T^*_2(a,0,\infty) = T^*_2(0,0,\infty) = \frac{\pi}{2}$. The three temperatures become very similar at $t^* = 2$. This means that the spot temperature becomes nearly uniform for $t^* \geq 2$.

4. Conclusions

Analytical exact solutions to calculate the thermal impedance and the transient temperature due to spot of heat on semi-infinite body are developed in this paper. The method is useful for physical applications such as laser beam, welding, electronic components junctions, thermal contacts between solids through asperities or others, rapid transient process (eg. Thermal projections), etc. These solutions do not necessitate any numerical computation, often difficult to obtain.

Numerical values are obtained rapidly, when the solutions are programmed using softwares, such as Maple or Mathematica for example. The method can be also extended to flux evolution other than those presented in this paper.

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